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Correlation between Weibull moduli for tensile and bending strength of brittle ceramics: a numerical simulation analysis based on a three-parameter Weibull distribution

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A numerical simulation was designed and performed to produce uniaxial tensile strength and three-point bending strength data. It was assumed that the specimen could be divided into many small units of volume for which the tensile strength followed a three-parameter Weibull distribution, characterized by the parameters m, σ_0 and σ_u. Statistical analysis of the strength data showed that the variation of the bending strength could be characterized by the explicit form of the three-parameter Weibull function as deduced from only the tensile test data. A strong correlation was found to exist between the Weibull modulus, $m_{E,B}$, estimated for the bending strength and that for tensile strength, $m_{E,T}$. The difference between $m_{E,B}$ and $m_{E,T}$ is dependent on m but independent of the ratio of σ_0 to σ_u.

1. Introduction

The fracture of brittle ceramics usually starts from defects preexisting in the material. As a consequence of the natural variability in size, location and severity of the preexisting defects, the measured fracture strength for a given ceramic usually shows a large scatter. The method most commonly used to characterize the statistical variation of the measured strength of ceramics is to apply Weibull statistical theory. The Weibull model assumes that there is a local strength associated with each small element of volume in a body. The risk of rupture for each element is integrated over the whole test specimen such that the probability of failure, P_f, is given as [1]

$$P_f = 1 - \exp\left[-\int_V \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m dV\right] \quad (1)$$

where σ is the stress at a point and V is the specimen volume. The Weibull parameters σ_u, σ_0 and m are the location parameter (or threshold strength), the scale parameter (or characteristic strength) and the shape modulus or more generally, the Weibull modulus, respectively.

Denoting σ_f as the maximum tensile stress that can exist in the test specimen, i.e., the fracture strength of the specimen, Equation 1 can be rewritten as

$$P_f = 1 - \exp\left[-V_e\left(\frac{\sigma_f - \sigma_u}{\sigma_0}\right)^m\right] \quad (2)$$

where V_e is the effective volume of the test specimen and can be expressed as

$$V_e = \int_V \left(\frac{\sigma - \sigma_u}{\sigma_f - \sigma_u}\right)^m dV \quad (3)$$

For specimens tested in uniaxial tension, σ is equal to σ_f and Equation 3 gives $V_e = V$. Thus Equation 2 can be simplified as

$$P_f = 1 - \exp\left[-\left(\frac{\sigma_f - \sigma_u}{MOR_0}\right)^m\right] \quad (4)$$

where $MOR_0 = \sigma_0 V_e^{-1/m}$ is a constant. By rearrangement, Equation 4 may be transformed as

$$\ln\left[\ln\left(\frac{1}{1 - P_f}\right)\right] = m\ln(\sigma_f - \sigma_u) - m\ln MOR_0 \quad (5)$$

Equation 5 shows that a plot of ln $\{\ln[1/(1 - P_f)]\}$ against $\ln(\sigma_f - \sigma_u)$ will give a straight line with a slope of m.

Since it is very difficult to perform a tensile test on brittle ceramics, [2], the fracture strength is generally measured in flexure, either using three-point bending or four-point bending [3]. Therefore, extension of the three-parameter Weibull distribution, Equation 1, to polyaxial stress states has been a interesting problem for a long time. Extensive work has been conducted to analyze the dependence of the Weibull modulus on the polyaxiality of the stress state [4–6] and some interesting conclusions have been obtained theoretically. As a completion to the previous theoretical studies, in the present study, a numerical simulation procedure was designed to produce three-point bending strength data for a ceramic for which the tensile strength follows a three-parameter Weibull distribution. Then the resulting bending strength data were analyzed to ascertain (i) whether or not the bending strength were characterized well by the explicit form of the three-parameter Weibull function, Equation 4, and (ii) whether or not the apparent three-parameter Weibull modulus resulting from the bend strength data could be correlated with its true value, i.e., the three-parameter Weibull modulus arising from the tensile strength data for the same material.

2. Numerical analysis procedure

2.1. Generation of the strength data

A computer program was designed to predict the bending strength of specimens, with traditional dimensions of 3 mm thick, 4 mm wide and 36 mm long, which were tested in three-point bending with a span of 30 mm. The material was assumed to have a tensile strength which followed a three-parameter Weibull distribution. The procedure of the simulation was as described below:

First, the volume under load of each bend test specimen was divided into 23040 ($= 12 \times 16 \times 120$) volume units with dimensions of $0.25 \times 0.25 \times 0.25$ mm (Fig. 1). Secondly, assuming that the tensile strength of each of these volume units could be described by Equation 4 with a set of prescribed values for the three parameters m, σ_u and MOR_0, a set of computer-generated random numbers in the range 0 to 1 were substituted for the fracture probability, P_f, in Equation 4 to calculate the fracture strength, $(\sigma_{f,U})_i$, for each volume unit. Then, using the coordinate values, x_i, y_i and z_i (see Fig. 1), for each unit, the applied bending force, F_i, which would result in a stress of $(\sigma_{f,U})_i$, in the corresponding unit was calculated according to the standard theory of mechanics of materials [7]. Finally, the bending strength, $\sigma_{f,B}$, of the specimen was calculated using the minimum among the resultant 23040 F_i-values and the specimen dimensions.

For each set of prescribed values of m, σ_u and MOR_0, a total of 30 specimens were "tested" according to the procedure described above to yield a *sample* distribution of bending strength. For each set of values, a total of 1000 samples, each containing 30 strength data, were generated. In all, 54 different sets of m, σ_u and MOR_0 values were examined, giving a grand total of 54,000 strength distributions and 162,000 individual values of strength. A similar procedure was adopted to obtain the tensile strength, $\sigma_{f,T}$, of "specimens" with the same dimensions, 3 mm thick, 4 mm wide and 36 mm long. To do this, the tensile strength for a given "specimen" was determined to be the minimum among resultant 23040 $(\sigma_{f,U})_i$-values. Similarly, for each set of prescribed values of m, σ_u and MOR_0, a total of 30 specimens were "tested" according to the procedure described above to yield a *sample* distribution of tensile strength and a total of 1000 samples, each containing 30 strength data, were generated for the purpose of statistical analysis.

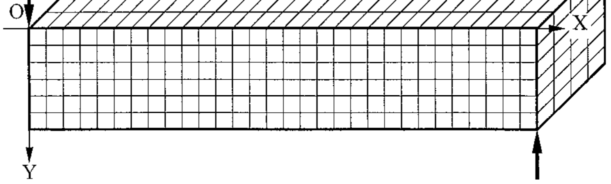

Figure 1 The three-point bend test specimen comprising 23040 volume units. Only one quarter of the specimen is shown.

2.2. Estimation of the Weibull modulus

For each sample of strength distribution data, the Weibull modulus was estimated using the traditional least-squares method. This was done by ordering the 30 data from the lowest to the highest. The i-th result in the set was assigned a cumulative probability of failure, P_i, [8–11]

$$P_i = \frac{i - 0.5}{n} \tag{6}$$

Note that the stress threshold σ_u for bend specimens is clearly determined by the lowest strength of the volume unit(s) subjected to the maximum bending moment. Therefore, it was assumed that the location parameter σ_u is independent of the test configuration and is constant, i.e., $\sigma_{u,B} = \sigma_{u,T} = \sigma_u$. Using the prescribed σ_u-value, a linear regression analysis was conducted, according to Equation 5, with the "measured" strengths and the corresponding probabilities to give the estimated Weibull parameter m_E. In the following sections, the estimated Weibull moduli are denoted as $m_{E,T}$ and $m_{E,B}$, where the subscripts "T" and "B" represent tension and bending, respectively.

3. Results and discussion

Fig. 2 shows examples of the Weibull plots, i.e., the plots of $\ln\{\ln[1/(1 - P_f)]\}$ against $\ln(\sigma_f - \sigma_u)$, of the "measured" tensile strength, $\sigma_{f,T}$, corresponding to different prescribed values of the three parameters m, σ_u and MOR_0. In constructing these plots, the P_f-value

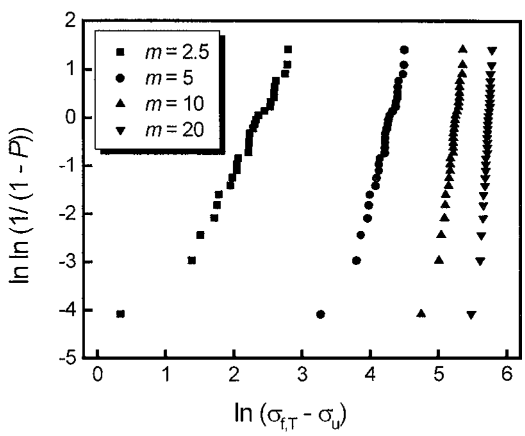

Figure 2 Weibull plots of the "measured" tensile strength, $\sigma_{f,T}$. The strength data were generated using $\sigma_u = 250$ MPa and $MOR_0 = 500$ MPa.

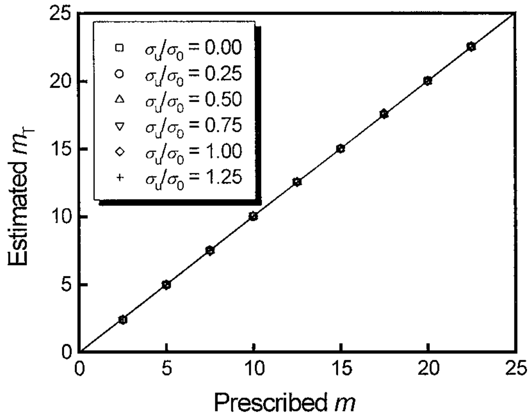

Figure 3 Plot of the estimated Weibull moduli from the tensile strength data as a function of the prescribed values of m, σ_u and MOR_0. The straight line in this figure is defined by $m_T = m$.

corresponding to each "specimen" was calculated using Equation 6. As can be seen, the linearity of the resultant Weibull plot is significant in each case.

Using the least-squares method described above, the Weibull modulus for each sample distribution of tensile strength was estimated and, for the 1000 samples corresponding to a given set of the prescribed values of m, σ_u and MOR_0, the average value, m_T, and the standard deviation, S_T, of the resultant 1000 $m_{E,T}$ were calculated. Fig. 3 shows m_T as functions of the prescribed m, σ_u and MOR_0. Clearly, all the data points in Fig. 3 fall along a straight line defined by $m_T = m$, indicating that the numerical simulation procedure designed and performed in the present study is reasonable. The coefficient of variation of $m_{E,T}$, i.e., the ratio of the standard deviation, S_T, to the average value, m_T, was found to be independent of the prescribed values of m, σ_u and MOR_0 and has a value of about 18%. Such a large scatter in $m_{E,T}$ is not surprising. In the previous studies [9, 10], the coefficient of variation of the estimated Weibull modulus for strength data which follow a two-parameter Weibull distribution has been reported to be sample-size-dependent and also has a typical value of about 18%.

The Weibull plots of the "measured" bending strengths, $\sigma_{f,B}$, corresponding to different prescribed values of the three parameters m, σ_u and MOR_0 are shown in Fig. 4. By comparing Fig. 4 with Fig. 2, it can be seen that, for a given set of prescribed values of the three parameters m, σ_u and MOR_0, the resultant bending strength is always somewhat higher than the tensile strength. The linearity of the resultant Weibull plot for bending strength is also significant. This finding may be very useful.

In general, it was suggested that Equation 4, the explicit form of the three-parameter Weibull function, is suitable only for analyzing the variation of tensile strength. Due to the non-uniformity of the stress distribution in bending specimens, it is very difficult, even impossible, to expand Equation 1 to a simple explicit form when studying the statistical properties of bending strength. Thus, in the past, authors [8, 12, 13] generally assumed $\sigma_u = 0$ for brittle ceramics and analyzed bending strengths according to the simplified, two-parameter Weibull function,

$$P_f = 1 - \exp\left[-V_e'\left(\frac{\sigma_{f,B}}{\sigma_0}\right)^m\right] \qquad (7)$$

where V_e' is the effective volume of the test specimen and can be treated as a constant for a given test configuration.

However, it should be pointed out that assuming $\sigma_u = 0$ seems to be questionable. As can be seen in Equation 1, the location parameter σ_u corresponds to a stress threshold below which the probability of failure for the test specimen would be zero. In other words, σ_u may be considered as the lowest strength for a given material. According to the classical Griffith criterion, the fact that a test specimen has a strength of zero means that the size of the preexisting defect in this specimen would be infinite. Undoubtedly, this is unreasonable. At least, such a situation cannot be encountered when the strength in bending is obtained from specimens with traditional dimensions of $3 \times 4 \times 36$ mm. Therefore, it is more appropriate to characterize the bending strength with a three-parameter Weibull distribution for a given ceramic. The numerical simulation results shown in Fig. 4 suggested that the bending strength of ceramics can also be characterized with Equation 4, making it possible and easy to analyze the bending strength with a three-parameter Weibull function.

Turning now to the next key issue. This is whether or not the apparent Weibull modulus, $m_{E,B}$, yielded by analyzing the bend strength data according to Equation 4 can be correlated with its true value, i.e., the three-parameter Weibull modulus, m or $m_{E,T}$, of the tensile strength for the same material.

Using the least squares method described above, the Weibull modulus for each sample distribution of bend-strength was estimated and, for the 1000 samples corresponding to a given set of the prescribed values of m, σ_u and MOR_0, the average value, m_B, and the standard deviation, S_B, of the resultant 1000 $m_{E,B}$ were calculated. The calculated m_B values are plotted as a function of m_T in Fig. 5, where each data point corresponds to a given set of prescribed values of m, σ_u and MOR_0.

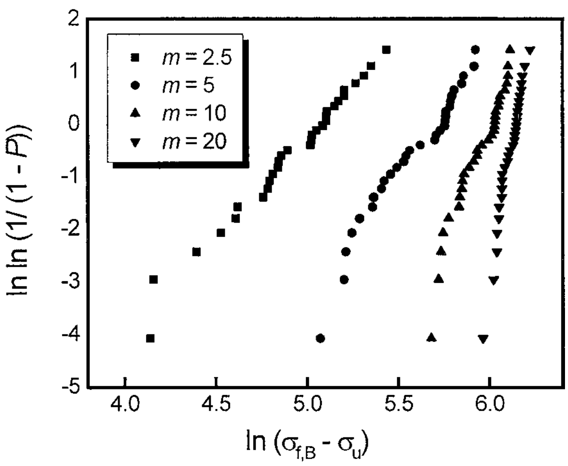

Figure 4 Weibull plots of the "measured" bending strength, $\sigma_{f,B}$. The strength data were generated using $\sigma_u = 250$ MPa and $MOR_0 = 500$ MPa.

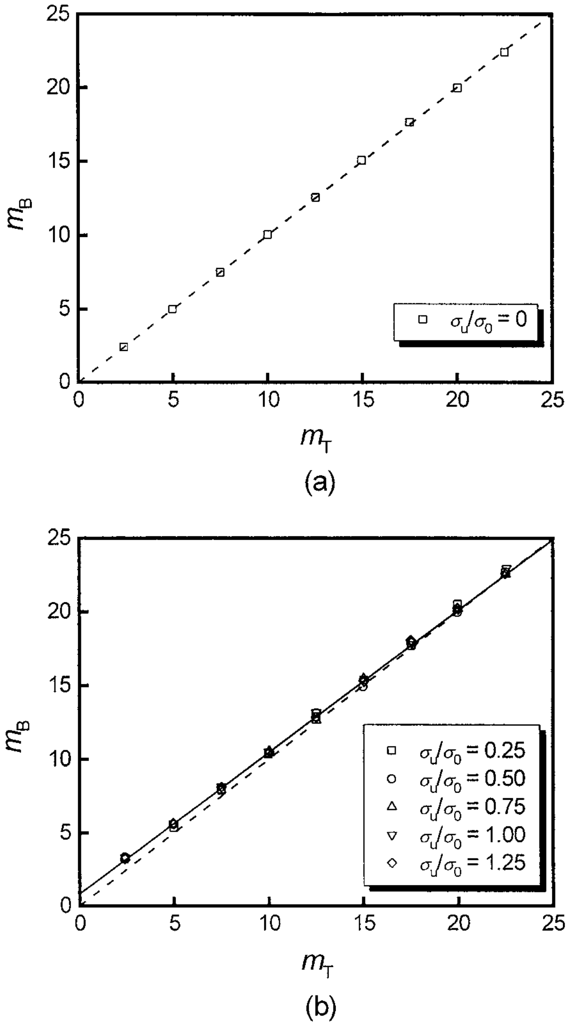

Figure 5 Relationship between m_B and m_T for (a) $\sigma_u/\sigma_0 = 0$ and (b) $\sigma_u/\sigma_0 \neq 0$. The dashed lines in these figures are defined by $m_B = m_T$.

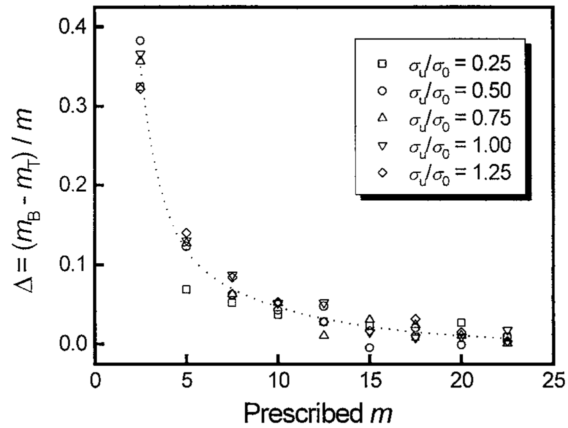

Figure 6 The difference between m_B and m_T as a function of the prescribed value of m.

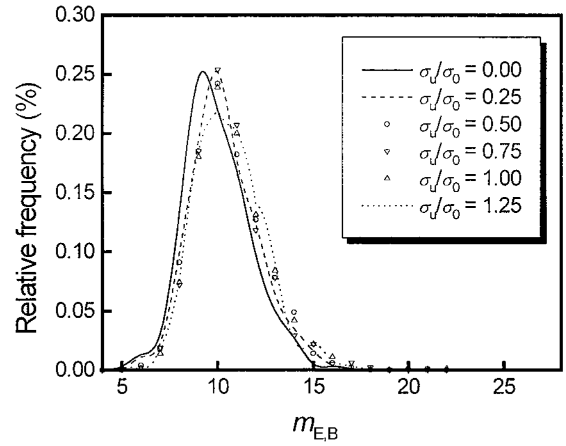

Figure 7 Probability density functions of the estimated $m_{E,B}$ for different prescribed σ_u/σ_0 ratios and a fixed prescribed m of 10.

As can be seen in Fig. 5a, all the data points corresponding to $\sigma_u/\sigma_0 = 0$, i.e., $\sigma_u = 0$, fall along a straight line defined as $m_B = m_T$, indicating that the Weibull modulus resulting from the bend strengths would be identical with that from the tensile strengths if the strength is assumed to follow a two-parameter Weibull distribution.

However, as shown in Fig. 5b, a slight difference between m_B and m_T was observed when the prescribed value of σ_u is not equal to zero. Such a difference, defined as $\Delta = (m_B - m_T)/m$, is now plotted as a function of the prescribed m in Fig. 6. Although each set of the five data corresponding to a given m shows a somewhat large scatter, it is reasonable to conclude from Fig. 6 that the difference between m_B and m_T is independent of the ratio of σ_u to σ_0. A similar conclusion can be obtained by comparing the empirical probability density functions (PDF) of $m_{E,B}$. Fig. 7 shows the PDFs of $m_{E,B}$ corresponding to different σ_u/σ_0 ratios. Clearly, it can be concluded that to a first approximation that there is no statistical difference between the $m_{E,B}$-values obtained for different σ_u/σ_0 ratios except for $\sigma_u/\sigma_0 = 0$.

Fig. 6 also indicates that Δ is strongly dependent on and decreases with the prescribed m. Such an experimental phenomenon can be understood easily by noting that the Weibull modulus, m, is a measure of the scatter of strength variation and a larger m would lead to a lower dispersion of fracture strengths.

A further comment should be made on the large scatter in Δ corresponding to a given m. In general, there are some uncertainties in the estimated values of the Weibull parameters associated with the inherent statistical fluctuations due to taking small sample sizes. In the present study, m_B corresponding to a given set of prescribed values of m, MOR_0 and σ_u exhibits a coefficient of variation of about 18%. This would make m_B-values obtained from different samples of 1000 estimated $m_{E,B}$ different from each other. A similar phenomenon was observed when analyzing the statistical variation of m_T. Note that the above analysis has shown that m_B is independent of the ratio of σ_u to σ_0. Therefore, the five m_B-data corresponding to different σ_u/σ_0 ratios can be considered to a small statistical sample and, thus, it seems not unreasonable to expect the large scatter in the Δ-values.

The results shown in Figs 5b and 6 make it possible to correlate the estimated m_B with its true value, m, with a simple equation in which the other two parameters, σ_u and σ_0, are excluded. As shown in Fig. 5b, a good linear relationship between m_B and m_T exists and this linear relationship can be expressed as

$$m_B = 0.80 + 0.97 m_T \tag{8}$$

Note that, as shown in Fig. 3, $m_T \approx m$. Therefore, Equation 8 can be used as an empirical equation to correct the estimated m_B to its true value. Of course, this empirical equation is suitable only for the sample distribution of 30 individual strength data.

It is of interest to make a brief comparison between the numerical simulation results obtained in the present study with the previously reported theoretical analysis. Ichikawa [6] has shown theoretically that, when the tensile strength of a brittle material follows a three-parameter Weibull distribution with a Weibull modulus of m, the three-point bending strength also follows a three-parameter Weibull distribution but with a Weibull modulus of $(m+2)$. The result of numerical simulation presented here is different from this theoretical prediction. This difference seems not to be surprising. In his work, Ishikawa considered directly the true distribution parameters, i.e., the distribution parameters for a sample of infinite size. The present study, however, concerns only small samples of 30 strength data. From a statistical viewpoint, there will generally be some systematic variability in the parameters estimated from a small sample due to inherent statistical fluctuations. In practice, it is only possible to actually test a limited number of specimens to obtain a sample distribution of strength data. Consequently, m can only be estimated with a sample of limited size. Thus the numerical simulation results obtained in the present study seem to be more useful than Ichikawa's work from the practical viewpoint.

It should also be pointed out that, in this study, an arithmetical averaging was used as the measure of the distribution of the estimated Weibull moduli. Similar treatment was also employed by other authors [9–11] in studying the statistical properties of the estimated Weibull modulus based on a two-parameter Weibull function. Undoubtedly, different conclusions would be yielded if other measures, such as the geometric mean or the 50%-ile value, were considered. Note that the estimated Weibull modulus does not follow a normal distribution [14, 15]. Further study should be conducted to check the efficacy of the arithmetical average in describing the statistical properties of the estimated Weibull modulus.

4. Concluding remarks

The numerical simulation analysis conducted in the present study suggests that the explicit form of the three-parameter Weibull function, Equation 4, can be used to characterize the statistical variation of the bending strength of a ceramic for which the tensile strength follows a three-parameter Weibull distribution. This conclusion is very useful because the strength of brittle ceramics is usually evaluated by bend testing rather than tensile testing and, undoubtedly, analysis of the bending strength based on a three-parameter Weibull function would be more reasonable than that based on a two-parameter Weibull function. However, it should be kept in mind that the Weibull modulus estimated by analyzing the bending strength data according to Equation 4 may be higher than its true value and should be corrected using an empirical relationship between m_B and m_T such as Equation 8.

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